

TWO FUNCTORS FROM ABELIAN GROUPS TO PERFECT GROUPS

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1. Introduction

The past decade or so has seen considerable development of the idea, originating with Kan and Thurston [12], of modelling a given sequence of abelian groups by the homology of another group. At the heart of such procedures lies the notion of an *acyclic group*, one having the same homology as the trivial group. (In this note all homology is taken to have trivial integer coefficients.) At some stage one seeks to embed a given group in a convenient acyclic group. Particularly striking in this context is the result of Baumslag, Dyer and Heller that when the given group G is abelian there exists an acyclic group of which G is the centre [1 Theorem 7.1]. Unfortunately the proof gives hardly any idea of what such an acyclic group must look like. Indeed, the proof is arguably ‘back-to-front’: it combines the existence of an Eilenberg–MacLane space of type $K(G, 2)$ with the main result of [12] in order to deduce the existence of the acyclic group. Here we stay more within the realm of group theory and give an explicit construction of an acyclic group with centre G . This construction involves the perfect locally nilpotent groups of McLain. Reversing the viewpoint of [1] gives a consequent construction of a $K(G, 2)$ space, differing from the usual model. We also obtain an explicit perfect group having G as its Schur multiplier; again, the existence, but not direct construction, of such a group is implied by [2 Theorem H]. An attraction of the constructions given below is their functorial nature.

To be specific, let Ab, Ac and Perf refer respectively to the categories of abelian, acyclic and perfect groups and their group homomorphisms. (Of course, since the first homology of a group is its abelianisation, Ac is a subcategory of Perf.) In Section 2 we define

$$A : \underline{\mathbf{Ab}} \rightarrow \underline{\mathbf{Ac}}$$

such that for an arbitrary abelian group G we have:

1.1. Theorem. *An abelian group G is naturally isomorphic to the centre $Z(\mathbf{A}(G))$ of the acyclic group $\mathbf{A}(G)$.*

We then define

$$\mathbf{B} : \mathbf{Ab} \rightarrow \mathbf{Perf} \quad \text{by} \quad \mathbf{B}(G) = \mathbf{A}(G)/Z(\mathbf{A}(G))$$

with the following, functorial, consequences.

1.2. Corollary. *$G \mapsto \mathbf{A}(G) \rightarrow \mathbf{B}(G)$ is the universal central extension of the perfect group $\mathbf{B}(G)$.*

1.3. Corollary. *G is isomorphic to the Schur multiplier $H_2(\mathbf{B}(G); \mathbb{Z})$ of the perfect group $\mathbf{B}(G)$.*

1.4. Corollary. *$K(\mathbf{B}(G), 1)^+ \simeq K(G, 2)$.*

Here $+$ denotes Quillen's plus-construction, as employed in, for example, [12] and [3]. It would be very nice indeed to see an iteration of (1.4), giving an explicit group-theoretic construction of all Eilenberg–MacLane spaces.

1.5. We investigate the question of how many cells must be adjoined to $K(\mathbf{B}(G), 1)$ in order to convert it to $K(G, 2)$ by means of the plus-construction as in (1.4) above. Recall (from, e.g., [3, p. 44]) that the plus-construction adjoins a 2- and 3-cell for each normally generating element of the perfect radical of the fundamental group. Thus the case of algebraic K -theory of a ring R requires only a single 2- and 3-cell [3, (9.6)] to convert BGLR to BGLR^+ . Here however the situation is markedly different.

1.6. Proposition. *The adjunction of cells to $K(\mathbf{B}(G), 1)$ to form $K(\mathbf{B}(G), 1)^+$ requires an infinite family of 2- and 3-cells.*

Incidentally, even in the case of infinite cyclic G , $K(\mathbf{B}(G), 1)^+$ gives a very different model for $K(G, 2)$ from the usual, infinite-dimensional complex projective space model, for the standard cell structure on $\mathbb{C}P^\infty$ contains no odd-dimensional cells.

1.7. (1.1) and (1.3) show that acyclic and perfect groups are too numerous even to form a set. (This also follows from the fact that algebraically closed groups are acyclic [1].) However, among non-trivial finite groups, acyclic groups are non-existent [18], while perfect groups appear to be rare. The latter comment may be quantified by means of the following.

1.8. Conjecture. *Let p_n, g_n denote the number of isomorphism classes of perfect,*

resp. all, groups of order n . Then

$$\text{dir lim } p_n/g_n = 0$$

where the limit is taken with respect to a suitable directed set structure on the natural numbers.

1.9. If one tries to broaden the constructions above to make them applicable to a wider class of groups than abelian groups, then serious problems are encountered. For instance, in (3.7) below we give examples of a group N which cannot be a normal subgroup of a perfect group P so that P acts trivially on N_{ab} . The general question

Which groups can be normal subgroups of acyclic groups?

is one to which we intend to return in a subsequent work. At the present time it is already clear that the class of such ‘normal-in-acyclic’ groups is highly circumscribed. What makes this remarkable is the fact [12] that every group may be embedded subnormally in an acyclic group with subnormal length at most 2.

2. Definitions

The key to our construction is a modification of a McLain group (itself a generalised unitriangular matrix group), whose definition we recall from [13], [15]. Let Λ be a linearly ordered set, F a field, and V a vector space over F with basis elements v_λ indexed by Λ . Then the McLain group $M(\Lambda, F)$ is the subgroup of the group of all linear transformations of V generated by elements of form $x_{\lambda\mu}^a$ where $a \in F$ and $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Here $x_{\lambda\mu}^a$ takes v_μ to $av_\lambda + v_\mu$ and fixes all other basis elements. For our purposes, it is more convenient to give an alternative description of $M(\Lambda, F)$ by means of a group presentation as in [7], [8]. We replace F by an arbitrary ring R .

2.1. Lemma. *The group $M(\Lambda, R)$ has presentation given by:*

generators

$$x_{\lambda\mu}^a, \quad a \in R; \quad \lambda, \mu \in \Lambda \text{ with } \lambda < \mu.$$

relations

$$x_{\lambda\mu}^a x_{\lambda\mu}^b = x_{\lambda\mu}^{a+b}, \tag{i}$$

$$[x_{\lambda\mu}^a, x_{\zeta\eta}^b] = \begin{cases} 1, & \mu \neq \zeta, \lambda \neq \eta, \\ x_{\lambda\eta}^{ab}, & \mu = \zeta. \end{cases} \tag{ii}$$

$$\tag{iii}$$

2.2. Definition. We modify (2.1) under the assumption (to hold henceforth) that the

ordered set Λ admits an initial element (normally written 0), and that N is a (unital) right R -module. Then the group $M(\Lambda, (R, N))$ is defined by the presentation (2.1) except that when $\lambda = 0$, $x_{0\mu}^a$ has $a \in N$ instead of $a \in R$. The relations are formally unchanged from (2.1) but should now be interpreted in terms of N when $\lambda = 0$.

2.3. Definition. Within a fixed group $M(\Lambda, (R, N))$ we may simplify notation for subgroups by letting $M(\Lambda_k)$ denote, for a given finite subbordering $\Lambda_k = \{\lambda_0 < \lambda_1 < \dots < \lambda_k\} \subset \Lambda$, the subgroup based on Λ_k (that is, generated by elements $x_{\lambda_i \lambda_j}^a$, $i < j$). The fact that $M(\Lambda, (R, N))$ is the direct limit of such subgroups will be exploited in Section 3 below.

2.4. Lemma. Suppose that Λ admits both first and last elements (labelled 0 and 1). Then the right R -module N is canonically isomorphic as an abelian group to the centre $Z(M)$ of $M = M(\Lambda, (R, N))$ via $a \mapsto x_{01}^a$ ($a \in N$).

Proof. The centre of a subgroup $M(\Lambda_k)$ based on $\Lambda_k = \{\lambda_0 < \dots < \lambda_k\}$ in Λ comprises all $x_{\lambda_0 \lambda_k}^a$ ($a \in R$ if $\lambda_0 > 0$, $a \in N$ if $\lambda_0 = 0$). However, if $\lambda_0 > 0$, then $x_{\lambda_0 \lambda_k}^1$ fails to centralise elements of form $x_{0\lambda_0}^a$ ($a \in N$). Similarly if $\lambda_k < 1$. This leaves the centre of M as consisting precisely of elements x_{01}^a ($a \in N$). So $a \mapsto x_{01}^a$ is the required isomorphism.

2.5. Definitions. Fix a dense linear ordering Λ with first and last elements 0 and 1. Dense means that whenever $\lambda < \mu$ in Λ then there exists $\nu \in \Lambda$ with $\lambda < \nu < \mu$. Then, using the standard (right) \mathbb{Z} -module structure on an arbitrary abelian group G , define

$$(i) \quad \mathbf{A}G = M(\Lambda, (\mathbb{Z}, G)),$$

$$(ii) \quad \mathbf{B}G = \mathbf{A}G / Z(\mathbf{A}G).$$

It is apparent that \mathbf{A} and \mathbf{B} define covariant functors on the category \mathbf{Ab} of abelian groups and group homomorphisms. The following alternative description of \mathbf{B} is a consequence of Lemma 2.4. It uses the notations $\hat{\Lambda} = \Lambda - \{0, 1\} \subset \Lambda$ and $G^{\hat{\Lambda}}, \mathbb{Z}^{\hat{\Lambda}}$, where the latter terms denote the set of all finitely supported functions from $\hat{\Lambda}$ to G (resp. \mathbb{Z}) – that is, sending all but finitely many elements to zero –, with abelian group (resp. ring) structure inherited from the codomain.

2.6. Proposition. $\mathbf{B}G$ is naturally isomorphic to the split extension

$$(G^{\hat{\Lambda}} \times \mathbb{Z}^{\hat{\Lambda}}) \rtimes M(\hat{\Lambda}, \mathbb{Z})$$

of the abelian group $G^{\hat{\Lambda}} \times \mathbb{Z}^{\hat{\Lambda}}$ by the acyclic group $M(\hat{\Lambda}, \mathbb{Z})$, where $x_{\lambda\mu}^n \in M(\hat{\Lambda}, \mathbb{Z})$ acts on $G^{\hat{\Lambda}} \times \mathbb{Z}^{\hat{\Lambda}}$ by means of:

$$[f, x_{\lambda\mu}^n](\nu) = \begin{cases} 0, & \nu \neq \mu, \\ nf(\lambda), & \nu = \mu, \end{cases} \quad f \in G^{\hat{\Lambda}};$$

$$[x_{\lambda\mu}^n, g](v) = \begin{cases} 0, & v \neq \lambda, \\ ng(\mu), & v = \lambda, \end{cases} \quad g \in \mathbb{Z}^{\hat{\Lambda}}.$$

2.7. It follows from (2.6) and (1.4) that the fibration

$$K(G^{\hat{\Lambda}} \times \mathbb{Z}^{\hat{\Lambda}}, 1) \rightarrow K(\mathbf{BG}, 1) \rightarrow K(M(\hat{\Lambda}, \mathbb{Z}), 1)$$

is an example of a fibration whose fundamental group homomorphism preserves perfect radicals but which fails to be plus-constructive [5], [6].

Of course in the above constructions \mathbb{Z} could be replaced by any ring over which a particular G is a module. The set Λ can range over all dense order types. Thus the possible isomorphism classes of groups \mathbf{AG} , \mathbf{BG} for fixed G are too numerous even to form a set.

3. Proofs

Our proof of the acyclicity of $M(\Lambda, (R, N))$ rests on the following lemma of [3, (3.11)] (see also [9, §1]).

3.1. Lemma. *Let $\varrho : M \rightarrow M$ be an endomorphism of a group M which is the direct limit of subgroups M_α , $\alpha \in A$, such that $\varrho(M_\alpha) \leq M_\alpha$. Suppose that for each $\alpha \in A$ there exist $\beta = \beta_\alpha$ (with $\alpha \leq \beta$), an element $y \in M_\beta$ and a homomorphism $\phi : M_\alpha \rightarrow C_{M_\beta}(M_\alpha)$ such that, for all $x \in M_\alpha$,*

$$x \cdot \phi(x) = y \cdot \varrho(x) \cdot \phi(x) \cdot y^{-1}.$$

Then $H_(\varrho)$ is an isomorphism, thus, in particular, if $\varrho^2 = \varrho$, then*

$$H_*(\varrho) = \text{id} : H_*(M; \mathbb{Z}) \rightarrow H_*(M; \mathbb{Z}).$$

Let a ring R and right R -module N be given, and remain fixed throughout the sequel. We apply (3.1) in the following way. For an ordering Λ with initial element τ , let ϱ_τ denote the composition

$$\varrho_\tau : M(\Lambda, (R, N)) \rightarrow M(\Lambda, (R, 0)) \rightarrow M(\Lambda, (R, N))$$

induced (by naturality) from the canonical R -module maps $N \rightarrow 0 \rightarrow N$. It is evidently an idempotent endomorphism.

3.2. Lemma. *Let Λ be a dense ordering with initial element τ . Then*

$$H_*(\varrho_\tau) = \text{id} : H_*(M(\Lambda, (R, N)); \mathbb{Z}) \rightarrow H_*(M(\Lambda, (R, N)); \mathbb{Z}).$$

Proof. Here $M = M(\Lambda, (R, N))$ is the direct limit of all subgroups of the form $M(\Lambda_k)$ where $\Lambda_k = \{\tau = \lambda_0 < \lambda_1 < \dots < \lambda_k\}$ (see (2.3)). Given Λ_k , let

$$\Lambda_{2k} = \{\tau = \lambda_0 < \lambda'_1 < \lambda_1 < \cdots < \lambda'_k < \lambda_k\} \subset \Lambda$$

for suitable $\lambda'_1, \dots, \lambda'_k \in \Lambda$. We define $y \in M(\Lambda_{2k})$ and $\phi : M(\Lambda_k) \rightarrow C_{M(\Lambda_{2k})}(M(\Lambda_k))$ with the required property. (For legibility we omit λ from the notation, so that $x_{h'h}^{-1}$ denotes $x_{\lambda_h \lambda_h}^{-1}$, etc.) Let $y = \prod_{h=1}^k x_{h'h}^{-1}$ (where all terms in the product commute), and, on taking $\lambda'_0 = \lambda_0$, put $\phi(x_{ij}^a) = x_{i'j'}^a$. Since $\varrho_\tau(x_{ij}^a) = x_{ij}^{a(1-\delta)}$ (where δ is the Kronecker delta δ_{i0}), the formula

$$x_{ij}^a x_{i'j'}^a = \left(\prod_{h=1}^k x_{h'h}^{-1} \right) x_{ij}^{a(1-\delta)} x_{i'j'}^a \left(\prod_{h=1}^k x_{h'h}^{-1} \right)$$

is easily verified from (2.2); for, the right-hand side reduces to

$$\begin{aligned} & (x_{ij}^{a(1-\delta)} x_{i'i}^{-1} [x_{i'i}^{-1}, x_{ij}^{a(\delta-1)}] x_{j'j}^{-1}) (x_{i'i}^1 [x_{i'j'}^a, x_{j'j}^1] x_{j'j}^1 x_{i'j'}^a) \\ &= x_{ij}^{a(1-\delta)} x_{i'i}^{-1} x_{i'j}^{a(\delta-1)} x_{i'i}^1 x_{i'j}^a x_{i'j'}^a \\ &= x_{ij}^{a(1-\delta)} x_{i'j}^{a\delta} x_{i'j'}^a, \end{aligned}$$

as it should.

We now derive a much stronger conclusion, by combining (3.2) with the fact that homology preserves direct limits. To show that M has trivial homology, it thus suffices to show that each inclusion of the form $i_k : M(\Lambda_k) \hookrightarrow M$ (where $\Lambda_k = \{\lambda_0 < \cdots < \lambda_k\}$) is homologically trivial. However, the inclusion $\Lambda_k \hookrightarrow \Lambda$ factors through the dense subordering $\Lambda_{\lambda_0} = \{\lambda \in \Lambda \mid \lambda \geq \lambda_0\}$ of Λ . Let M_{λ_0} denote $M(\Lambda_{\lambda_0}, (R, N))$ or $M(\Lambda_{\lambda_0}, R)$ according as λ_0 is or is not minimal in Λ . Then by (3.2) we deduce that $H_*(i_k) = H_*(i_k) \circ H_*(\varrho'_{\lambda_0})$ where $\varrho'_{\lambda_0} : M(\Lambda_k) \rightarrow M(\Lambda_k)$ denotes $\varrho_{\lambda_0} : M_{\lambda_0} \rightarrow M_{\lambda_0}$ restricted to $M(\Lambda_k)$. Since ϱ'_{λ_0} is the composition $M(\Lambda_k) \rightarrow M(\Lambda_k) \hookrightarrow M(\Lambda_k)$ where ${}_h\Lambda_k = \{\lambda_h < \cdots < \lambda_k\}$ (with ${}_h i_k : M({}_h\Lambda_k) \hookrightarrow M(\Lambda_k) \hookrightarrow M$) we may in turn apply this deduction to $M({}_1\Lambda_k)$, etc. Iteration therefore yields that

$$\begin{aligned} H_*(i_k) &= H_*({}_1 i_k) \circ H_*(\varrho'_{\lambda_1}) \circ H_*(\varrho'_{\lambda_0}) \\ &= H_*({}_2 i_k) \circ H_*(\varrho'_{\lambda_2} \circ \varrho'_{\lambda_1} \circ \varrho'_{\lambda_0}) \\ &= \cdots = 0, \end{aligned}$$

because the composition $\varrho'_{\lambda_{k-1}} \circ \cdots \circ \varrho'_{\lambda_0}$ is trivial. So M is itself homologically trivial after all.

3.3. Proposition. *For any dense ordering Λ with initial element the group $M = (\Lambda, (R, N))$ is acyclic.*

Of course, when $N=R$ it should be clear from our argument that the condition that Λ have an initial element is no longer needed. The result for this case (kindly pointed out to me by J.B. Wagoner) is due to Suslin [16], [17], whose proof is essentially as above. The question as to whether McLain groups which are perfect

(equivalently, which are defined on dense orderings) must also be acyclic was first posed in [8].

3.4. (1.2) is immediate from (1.1) and standard facts about universal central extensions of perfect groups (e.g. [14, Ch. 5], [3, Ch. 8]) because $\mathbf{A}(G)$ is superperfect. Again, it is standard that for such an extension the kernel is the Schur multiplier. From this central extension we deduce that the fibration

$$K(G, 1) \rightarrow K(\mathbf{A}(G), 1) \rightarrow K(\mathbf{B}(G), 1)$$

is plus-constructive [11], [3, (8.4)], [4, (1.1)], that is, that there is also a fibration

$$K(G, 1) \rightarrow K(\mathbf{A}(G), 1)^+ \rightarrow K(\mathbf{B}(G), 1)^+.$$

Since $\mathbf{A}(G)$ is acyclic, $K(\mathbf{A}(G), 1)^+$ is contractible, making $\Omega K(\mathbf{B}(G), 1)^+ \simeq K(G, 1)$ as required for (1.4) (cf. [19], which uses a similar argument in order to exhibit $BGLR^+$ as a loop space for a ring R).

3.5. To prove (1.6) we show that no finitely generated subgroup L of $\mathbf{B}(G)$ can have $\mathbf{B}(G)$ as its normal closure. This will be an application of the following lemma to the semi-direct product structure of $\mathbf{B}(G)$ exhibited in Proposition 2.6.

3.6. Lemma. *Let KM be a semi-direct product of the normal subgroup K by the subgroup M , and let L be a subgroup of KM . Then*

- (a) $L = (K \cap L)(KL \cap M)$.
- (b) *If $L = L_1 L_2$ with $L_1 \leq K$ and $L_2 \leq M$, then $L_1 = K \cap L$.*
- (c) *The normal closure L^{KM} of L in KM is given by*

$$L^{KM} = (K \cap L)^{KM} [K, KL \cap M] (KL \cap M)^M.$$
- (d) *For abelian K , $K \cap (L^{KM}) = (K \cap L)^M [K, L]$.*

Proof. (a) If we use the obvious isomorphism of M with KM/K , then taking the quotient of both sides of our equation by the normal subgroup $K \cap L$ reduces the problem to showing that this isomorphism identifies $KL \cap M$ with KL/K . However, by Dedekind's Rule,

$$KL = KL \cap KM \cong K(KL \cap M),$$

so that factoring out K gives the result.

(b) This is a straightforward application of Dedekind's Rule.

(c) From the standard fact that $L^{KM} = L[KM, L]$ where $[KM, L]$ is normal in KM , we have after (a)

$$L^{KM} (K \cap L) [KM, L] (KL \cap M) = (K \cap L) [KM, (K \cap L)(KL \cap M)] (KL \cap M),$$

which assumes the required form when the commutator is expanded.

(d) This uses (b) and (c), together with the remark that, when K is abelian,

$$[K, KL \cap M] \leq [K, KL] = [K, L] \leq K \cap L^{KM}.$$

Now suppose L to be a non-trivial finitely generated subgroup of $\mathbf{B}(G)$. From the observation that any element of $M = M(\hat{A}, \mathbb{Z})$ can be written in the form

$$\prod_{\mu} x_{\lambda_0 \mu}^{a_{\mu}} \prod_{\substack{\lambda > \lambda_0 \\ \mu}} x_{\lambda \mu}^{b_{\lambda \mu}}$$

for some uniquely determined $\lambda_0 \in \hat{A}$, we obtain a function $\varrho_1 : M \rightarrow \hat{A}$ sending each element to its corresponding λ_0 . Likewise, there is a surjection $\varrho_2 : G^{\hat{A}} \rightarrow \hat{A} - \{0\}$ sending each non-trivial locally finite $f : \hat{A} \rightarrow G$ to the minimum v for which $f(v) \neq 0$ and taking the zero map to 1. Then for any non-trivial element of $G^{\hat{A}}M$ in $\mathbf{B}(G)$ we define ϱ to be the minimum of its associated ϱ_1 and ϱ_2 . Note that, from (2.6), ϱ is non-decreasing on products and increasing on commutators in $[G^{\hat{A}}, M]$. Now let $\delta \in \hat{A}$ be minimal among all values of ϱ obtained in this way from the finite set of generators of L . Then from (3.6)(d) $\varrho(G^{\hat{A}} \cap L^{\mathbf{B}(G)}) \subset [\delta, 1]$, so that, because $\varrho = \varrho_2$ is surjective on $G^{\hat{A}}$, $L^{\mathbf{B}(G)} \neq \mathbf{B}(G)$.

3.7. The following class of examples illustrates the difficulty of generalising our methods to non-abelian G . We use a family of groups considered by Dyer and Formanek in [10], namely those of form F/R' where F is a free group of finite rank at least 2, $R \leq F'$ is normal in F and F/R is residually torsion-free nilpotent. Let N admit a normal subgroup K with quotient N/K in the above family. Note that since N/K is centreless and residually torsion-free nilpotent [10, Theorem B] K will actually be characteristic if central or perfect or generated by elements of finite order (or a product of such subgroups).

When K is characteristic in N and N is normal in some perfect group P , we obtain an extension $N/K \twoheadrightarrow P/K \twoheadrightarrow P/N$. Since N/K is centreless, this extension is induced from the canonical extension $N/K \twoheadrightarrow \text{Aut}(N/K) \twoheadrightarrow \text{Out}(N/K)$ by some map $P/N \rightarrow \text{Out}(N/K)$. As p is perfect so too must be the image of such a map; therefore its image cannot lie in $\text{Ker}[\text{Out}(N/K) \rightarrow \text{Aut}((N/K)_{\text{ab}})]$, because by [10, Theorem C] that kernel is residually nilpotent. (Of course, the possibility that $P/N \rightarrow \text{Out}(N/K)$ be trivial is excluded, for that would imply that $P/K \cong N/K \times P/N$ where P/K is perfect but N/K is not.) In other words, the induced action on $(N/K)_{\text{ab}}$ must be non-trivial. Hence the action of P on N_{ab} cannot be trivial either.

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